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The algebriac theory of quasicrystals with five-fold symmetries*

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...And I saw the sacred hoop of my people was one of the many hoops that made one circle, wide as daylight and as starlight, and in the center grew one mighty flowering tree to shelter all the children of one mother and one father.

—Black Elk, Oglala Sioux

Abstract. An algebraic binary operation is introduced into quasicrystals admitting five-fold symmetry. In terms of this many quasicrystals displaying full pentagonal or icosahedral symmetry are seen to be finitely generated. Examples are given in dimensions 1, 2, 3 and 4. The operation of left quasicrystal addition is affine-linear. The monoid generated by these operators is discussed and a presentation for it given in the generic case.

1. Introduction

In this paper we wish to point out that a great variety of quasicrystals admitting five-fold symmetries are closed under an algebraic binary operation that we call *quasicrystal addition*. In terms of this operation many infinite quasicrystals displaying full pentagonal or icosahedral symmetry can be finitely generated.

For the purposes of this paper we mean by quasicrystals certain point sets in real n-space $\mathbb{R}^n$. Initially the concept is used loosely, since no generally accepted definition of the word exists. Our main requirement is that our sets Σ possess the *Delaunay property*: there exists positive constants r_1 and r_2 such that

$$\text{for all } x \in \Sigma \text{ the ball } B_x(r_1) \text{ of radius } r_1 \text{ about } x \text{ meets } \Sigma \text{ only in } x : \Sigma \cap B_x(r_1) = \{x\}, \text{ for all } x \in \Sigma \tag{1.1}$$

$$\text{for all } x \in \mathbb{R}^m,\ \Sigma \cap B_x(r_2) \neq \emptyset. \tag{1.2}$$

We let $\tau := (1+\sqrt{5})/2$, $F := \mathbb{Q}[\tau]$, and let $' : F \longrightarrow F$ be the automorphism determined by $\sqrt{5} \longmapsto -\sqrt{5}$. The subring $R := \mathbb{Z}[\tau] = \{a + b\tau \mid a, b \in \mathbb{Z}\}$ of F is the ring of integers of F. We recall that

$$\tau^2 = \tau + 1 \qquad \tau + \tau' = 1. \tag{1.3}$$

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An *R-lattice* in $\mathbb{R}^n$ is an R-submodule L of $\mathbb{R}^n$ of rank n that spans $\mathbb{R}^n$. All of our quasicrystals will consist of subsets of points from R-lattices. Quasicrystal addition, $\vdash$, is defined by

$$x \vdash y = \tau^2 x - \tau y.$$

What makes this relevant to quasicrystals is the close relationship to expressions of the form

$$\tau'^2 x - \tau' y$$

which are in fact *convex* combinations of x and y. We define quasilattices to be Delaunay subsets of R-lattices that are closed under $\vdash$. We provide a number of examples of such sets in dimensions 1, 2, 3 and 4 which show the ubiquity of such sets.

The affine-linear operators $T_x : y \longmapsto (x \vdash y)$ are particularly interesting. We view the monoid of these operators as a generalization to the quasicrystal setting of the group of translations of a lattice. The underlying identity that relates these operators is

$$T_x T_y T_y = T_y T_x T_x$$

which we think of as a replacement for commutativity. In section 6 we prove that if B is a base for the R-lattice L then these identities completely describe the monoid $\mathcal{T}_B$ generated by the T_x, $x \in B$.

In section 3 we establish a simple geometric condition that can be used to show that certain quasicrystals defined by acceptance domains are generated by finite sets of elements. In section 4 we examine the one-dimensional cases in detail.

The illustrations accompanying the text were generated by the software package *simpLie* [4].

2. Quasicrystal addition

Let L be a free R-module and let $L_F := F \otimes L$. In the case that L is an R-lattice, L_F will be considered simply as the F-span of L in $\mathbb{R}^n$.

We define *quasicrystal addition* on L_F by

$$x \vdash y := \tau^2 x - \tau y$$

for all $x, y \in L$. For each $x \in L_F$ we then have the operator

$$T_x : L_F \to L_F$$

$$y \longmapsto (x \vdash y) = \tau^2 x - \tau y.$$

Observe that T_x is an affine-linear mapping of L_F. Quasicrystal addition is neither associative nor commutative. However we have:

Proposition 2.1. For all $x, y, u \in L_F$:

(i) $T_x(x) = x \vdash x = x$;
(ii) $x \vdash (x \vdash y) = y \vdash x$;
(iii) $T_x T_y^2 = T_y T_x^2$;
(iv) $(x + u) \vdash (y + u) = (x \vdash y) + u$.

Proof. These are all trivial to verify. For (iii) we note that by using (1.3)

$$\begin{aligned} T_x T_y^2 z &= \tau^2 x - \tau(\tau^2 y - \tau(\tau^2 y - \tau z)) \\ &= \tau^2 x + (\tau^4 - \tau^3) y - \tau^3 z \\ &= \tau^2 (x + y) - \tau^3 z \end{aligned}$$

which is symmetric in x and y. □

The important property of proposition 2.1(iv) is called *translation invariance.*

A subset B of L_F is *closed* under $\vdash$ if $x \vdash y \in B$, for all $x, y \in B$. For non-empty subsets $A, B \subset L_F$ we define

$$(A; B)^{\vdash} := \{T_{a_k} \dots T_{a_1}(b) : a_1, \dots a_k \in A, b \in B, k \geqslant 0\}$$

where it is understood that we are taking the elements of B when $k = 0$. We call A the set of *active generators* and B the set of *passive generators* in $(A; B)^{\vdash}$. We define

$$A^{\vdash} = (A; A)^{\vdash}.$$

We say that A *replicates* a set B or B is *replicated by* A if

(i) $T_a(B) \subset B$, for all $a \in A$,

(ii) $\bigcup_{a \in A} T_a(B) = B$.

Figure 1 illustrates the growth of a set $(A; A)^{\vdash}$ where A is the set of vertices of a regular pentagon. Beginning with A we see successively $A \vdash A$, $A \vdash (A \vdash A)$, and $A \vdash (A \vdash (A \vdash A))$.

In order to understand how quasicrystal addition can be introduced naturally into the study of quasicrystals, it is necessary to formulate the cut and project method in terms of semilinear maps on L.

It is simplest to begin with the case $L = R$ and the automorphism $' : R \to R$. Since

$$R = \mathbb{Z} + \mathbb{Z}\tau \simeq \mathbb{Z} \times \mathbb{Z} \subset \mathbb{R} \times \mathbb{R}$$

R may be viewed either as a one-dimensional R-module or a two-dimensional $\mathbb{Z}$-module. In terms of the standard dot product on $\mathbb{R}^2$, the two vectors

$$c(1, \tau), c'(1, \tau') \qquad \text{where } c := (1 + \tau^2)^{-1/2}$$

form an orthonormal basis and we have the projection maps

$$\begin{aligned} (a, b) &\longmapsto c(a + \tau b) \\ (a, b) &\longmapsto c'(a + \tau' b). \end{aligned}$$

From our point of view, the scaling factors c and c' are irrelevant and we prefer to use

$$\pi_{\parallel} : (a, b) \longmapsto (a + \tau b)$$

and

$$\pi_{\perp} : (a, b) \longmapsto (a + \tau' b).$$

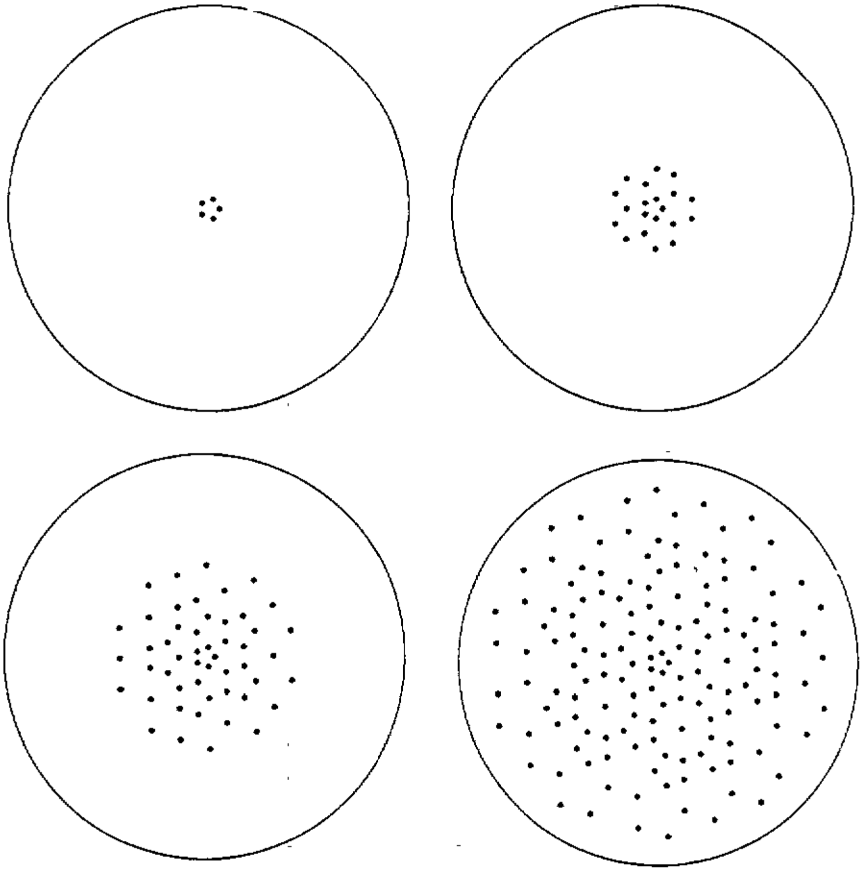

Figure 1. The sets A, $A \vdash A$, $A \vdash (A \vdash A)$, and $A \vdash (A \vdash (A \vdash A))$ illustrating the growth of $(A; A)$ where A is the set of vertices of a pentagon.

Thus the composite maps

$$R \simeq \mathbb{Z} \times \mathbb{Z} \subset \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$

where the last map is either $\pi_\parallel$ or $\pi_\perp$, are

$$a + b\tau \longmapsto a + b\tau$$

and

$$a + b\tau \longmapsto a + b\tau' = (a + b\tau)'$$

respectively. Thus $\pi_\parallel$ consists of viewing R as a subset of $\mathbb{R}$ and $\pi_\perp$ is then simply the automorphism $'$.

The essence of the cut and project method is to select a suitable point-set Σ in L ($= R$ in this case) by imposing a bounded set with non-empty interior as the target space of $\pi_\perp$.

Example 2.2. Let P be an interval (= convex subset) of $\mathbb{R}$. We define

$$\Sigma_P = \Sigma_P(R) = \{x \in R : x' \in P\}.$$

We claim that Σ_P is closed under $\vdash$. In fact, for $x, y \in \Sigma_P$,

$$(\tau^2 x - \tau y)' = (\tau')^2 x' - \tau' y' = \tau^{-2} x' + \tau^{-1} y'.$$

Since $0 < \tau^{-2}, \tau^{-1} < 1$ and $\tau^{-2} + \tau^{-1} = 1$, we have $(\tau^2 x - \tau y)' \in P$ by the convexity of P.

Observe also that $\tau \Sigma_P \subset \Sigma_P$ provided that $\tau' P \subset P$. If P is a finite interval with non-empty interior then Σ_P is a Delaunay set (see proposition 2.2 below).

To generalize this example we introduce the following algebraic structure:

Definition 2.3. An $(R,^*)$-module is a pair $\mathcal{L} = (L,^*)$ consisting of an R-lattice

$$L = \oplus_{j=1}^n R e_j \subset \mathbb{R}^n$$

together with a mapping

$$^* : L \longrightarrow \mathbb{R}^n$$

satisfying

$$(x + y)^* = x^* + y^*$$
$$(rx)^* = r' x^* \qquad \text{for all } x, y \in L,\ r \in R$$
$$L^* \text{ spans } \mathbb{R}^n.$$

Such a mapping is necessarily injective.

A particularly important instance of this is the pair

$$\mathcal{R} = (R,').$$

In many cases we have $L^* \subset L$, but there are important cases (see the icosian ring below) in which this does not happen. Nor need * be an operator of order 2, although this is often the case. We will assume in the following that $\mathbb{R}^n$ is equipped with a Euclidean norm $\| \quad \|$.

An easy way to construct a semilinear mapping * on L is to define it by

$$\sum_{j=1}^n a_j e_j \longmapsto \sum_{j=1}^n a_j' e_j.$$

We will continue to view L_F (and L and L') as subsets of $\mathbb{R}^n$.

Example 2.4. Let P be any subset of $\mathbb{R}^n$. We define

$$\Sigma_P = \Sigma_P(\mathcal{L}); = \{x \in L : x^* \in P\}.$$

Precisely the same type of argument that we used in example 2.1 shows that if P is convex then Σ_P is closed under $\vdash$. We call P the *acceptance domain* of the set Σ_P.

The constuction of Σ_P in example 2.4 can be interpreted as a form of the cut and project method, as we pointed out in the special case of example 2.2. Thus a large number of quasicrystals that appear in the literature can be accommodated by example 2.4.

Lemma 2.5. Let $\mathcal{L} = (L, ^*)$ be an $(R, ^*)$-module. Then:
(i) L and L^* are dense subsets of $\mathbb{R}^n$;
(ii) for any $r_1, r_2 > 0$,

$$\{x \in L : \|x\| < r_1, \|x^*\| < r_2\}$$

is finite. Furthermore, for small enough r_1 this set is reduced to $\{0\}$.

Proof. (i) follows from the fact that R is dense in $\mathbb{R}$. For (ii) we observe that for $x = \Sigma c_j e_j$, $c_j \in R$, the conditions $\|x\| < r_1$ and $\|x'\| < r_2$ simultaneously bound c_j and c'_j for each j. But the number of elements $a + b\tau$ with $a, b \in \mathbb{Z}$ satisfying $|a + b\tau| < M_1$ and $|a + b\tau'| < M_2$ is finite for all $M_1, M_2 > 0$. Furthermore, if M_1 is small enough the only solution to these inequalities is $a = b = 0$. □

Definition 2.6. An *quasilattice* is a subset Λ of an R-lattice in $\mathbb{R}^n$ satisfying:
(i) Λ is closed under $\vdash$;
(ii) Λ is a Delaunay set in $\mathbb{R}^n$.
The R-module K generated by Λ in L is called the *ambient space* of Λ.

Proposition 2.7. Let $\mathcal{L}$ be an $(R, ^*)$-module in $\mathbb{R}^n$ and let $P \subset \mathbb{R}^n$ be a bounded convex subset with non-empty interior. Then $\Sigma_P(\mathcal{L})$ is a quasilattice.

Proof. $\Sigma_P(\mathcal{L})$ is closed under $\vdash$ by (2.4). Let $s > 0$. There exists $r_1 = r_1(s) > 0$ such that $r_1 \to 0$ as $s \to 0$ and such that

$$\|\sum a_j e_j\| < s \Rightarrow | a_j | < r_1 \qquad \text{for all } j.$$

There exists r_2 such that

$$\sum a_j e_j^* \in 2P \Rightarrow | a_j | < r_2 \qquad \text{for all } j.$$

Let $x, y \in \Sigma_P(\mathcal{L})$. Write $x - y = \sum a_j e_j$, $a_j \in R$. Suppose that $\|x - y\| < s$. Then for all $j = 1, \ldots, n$

$$| a_j | < r_1 \qquad \text{and} \qquad | a'_j | < r_2.$$

Using lemma 2.5 we obtain (1.1) for $\Sigma_P(\mathcal{L})$.

Since L^* is dense and P has a non-empty interior, there exist $v_1, \ldots, v_n \in \Sigma_P(\mathcal{L})$ that span $\mathbb{R}^n$. Using proposition 4.4, which we will prove below, we see that $\{0, v_j\}^{\vdash}$ is a Delaunay set. Thus there is an $r_j > 0$ such that any interval of length at least r_j in $\mathbb{R}v_j$ contains a point of $\{0, v_j\}^{\vdash}$. Using this for all $j = 1, \ldots, n$ we obtain (1.2) for $\Sigma_P(\mathcal{L})$. □

Example 2.8. Inside the real quaternions

$$\mathbb{H} = \mathbb{R}1 + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$$

we consider the subring $\mathbb{H}_F = F1 + Fi + Fj + Fk$. we define * on $\mathbb{H}_F$ as the $(F,')$-semilinear mapping that is uniquely specified by $l^* = l$, $l \in \{1, i, j, k\}$. The 120 unit quaternions

$(\pm 1, 0, 0, 0)$	and all permutations
$\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$	
$\frac{1}{2}(0, \pm 1, \pm\tau', \pm\tau)$	and all even permutations

called *icosians* form a finite group I isomorphic to the binary icosahedral group. The $\mathbb{Z}$-span of I is a ring $\mathbb{I}$ called the *icosian ring*. The icosian ring is one of the (all conjugate) maximal orders of $\mathbb{H}_F$. The pair $\mathcal{I} = (\mathbb{I},^*)$ is an $(R,^*)$ module.

If P is any bounded convex subset of $\mathbb{H}$ with non-empty interior we may form the quasilattice $\Sigma_P(\mathcal{I}) = \{x \in \mathbb{I} \mid x^* \in P\}$. (See [3, 5] for more details when P is an open ball.)

There are any number of quasilattices that we may form here that possess remarkable symmetry but about which we know virtually nothing. For instance, the set I forms the vertices of the four-dimensional polytope $\{3, 3, 5\}$ [2]. The quasilattice $\Sigma_{\{3,3,5\}}(\mathcal{I})$ contains the quasilattices $I^{\vdash}$ and $\{I; I \cup \{0\}\}^{\vdash}$. How do they compare?

Similarly the set $\mathbb{I}_0 := \mathbb{I} \cap (\mathbb{R}i + \mathbb{R}j + \mathbb{R}k)$ gives rise to three-dimensional quasilattices. If S is a set of vertices with icosahedral symmetry then $S^{\vdash}$ and $\{S; S \cup \{0\}\}^{\vdash}$ are quasilattices with icosahedral symmetry. In general we know little about these. In the case that the vertices are the 30 (resp. 120) points of the root system of type H_3 (resp. H_4) we have a result analogous to the claims of propositions 5.1 and 5.3 [1].

Proposition 2.9. Let Λ be a quasilattice in the R-lattice L. Then the F-span of Λ is L_F and the ambient space is an R-lattice in L.

Proof. Since Λ is a Delaunay set, it cannot lie in a proper subspace of $\mathbb{R}^n$. Thus Λ contains a basis of $\mathbb{R}^n$ and hence also of L_F. It follows that the ambient space is a submodule of rank n. □

Definition 2.10. Let Λ_1, Λ_2 be quasilattices with ambient spaces L_1 and L_2 respectively. An *affine mapping* of L_1 into L_2 is a mapping $\phi : L_1 \longrightarrow L_2$ for which there is an R-module map $\partial\phi : L_1 \longrightarrow L_2$ such that

$$\phi(x + v) = \phi(x) + \partial\phi(v)$$

for all $x, v \in L$.

A *homomorphism* of Λ_1 into Λ_2 is a mapping $\phi : L_1 \longrightarrow L_2$ satisfying:

(i) ϕ is an afine map;

(ii) $\phi(\Lambda_1) \subset \Lambda_2$.

If $L_1 = L_2$ then it is an *endomorphism*. It is an *isomorphism* if ϕ is bijective and $\phi(\Lambda_1) = \Lambda_2$. We write $\Lambda_1 \simeq \Lambda_2$.

Let ϕ be a homomorphism from Λ_1 into Λ_2. Let $\phi(0) = a \in L_2$. Then

$$\phi(x) = \phi(0 + x) = a + \partial\phi(x) \qquad \text{for all } x \in L_1.$$

Then for all $x, y \in \Lambda_1$,

$$\begin{aligned}\phi(T_x(y)) &= \phi(x \vdash y) = \phi(\tau^2 x - \tau y)\\ &= a + \partial\phi(\tau^2 x - \tau y)\\ &= a + \tau^2\partial\phi(x) - \tau\partial\phi(y)\\ &= \phi(x) \vdash \phi(y) = T_{\phi(x)}(\phi(y)).\end{aligned}$$

Example 2.11. If Λ is a quasilattice in L then:
(1) $\Lambda \simeq \Lambda + v$, for all $v \in L$;
(2) for all x in Λ, T_x is an endomorphism of Λ.

Remark 2.12. If a quasilattice Λ is closed under central symmetry ($\Lambda = -\Lambda$) then we can define another binary operation

$$(x, y) \longmapsto \tau^2 x + \tau y$$

on Λ and obtain a new set of operators

$$S_x : y \longmapsto \tau^2 x + \tau y.$$

It is intuitively suggestive to think of these S_x's as quasicrystal variants (when τ'='1) of the operators

$$L_x : y \longmapsto x + y$$

that define the fundamental symmetries of lattices. Form this viewpoint, the monoid $\mathcal{M}$ generated by the operators S_x, $x \in \Lambda$ becomes the natural analogue of the group of translations of a lattice. The basic identity

$$S_x^2 S_y = S_y^2 S_x \qquad \text{for all } x, y \in \Lambda$$

then becomes the quasilattice analogue of the commutatitve law

$$L_x L_y = L_y L_x.$$

3. Replication

In this section we give a simple geometric condition for replication.

Let $\mathcal{L} = (L, *)$ be an $(R, *)$ module in $\mathbb{R}^n$ above. It is straightforward to verify:

Lemma 3.1. Let $P, Q \subset \mathbb{R}^n$. Then:
(i) $\Sigma_{\tau^k P} = (\tau')^k \Sigma_P$, for all $k \in \mathbb{Z}$;
(ii) $\Sigma_{\lambda^* + P} = \lambda + \Sigma_P$, for all $\lambda \in L$;
(iii) $\Sigma_P \cup \Sigma_Q = \Sigma_{P\cup Q}$, $\Sigma_P \cap \Sigma_Q = \Sigma_{P\cap Q}$. □

Proposition 3.2. Let P be a closed bounded convex subset of $\mathbb{R}^n$ with non-empty interior. Let S be a finite subset of L. Then S is replicating for Σ_P if and only if

$$\tau P = \bigcup_{s \in S} (\tau^{-1} s^* + P). \tag{RC}$$

Proof. S replicates Σ_P

$$\begin{aligned}
&\iff \Sigma_P = \bigcup_{s\in S} T_s \Sigma_P \\
&\iff \Sigma_P = \bigcup_{s\in S} (\tau^2 s - \tau \Sigma_P) \\
&\iff -\tau^{-1}\Sigma_P = \bigcup_{s\in S} (-\tau s + \Sigma_P) \\
&\iff \Sigma_{\tau P} = \bigcup_{s\in S} \Sigma_{\tau^{-1}s^* + P} = \Sigma_{\bigcup(\tau^{-1}s^*+P)}
\end{aligned}$$

using lemma 3.1 several times.

If S is replicating for Σ_P then τP and $\bigcup(\tau^{-1}s^* + P)$ are closed regions in $\mathbb{R}^n$ that determine the same set of points in L. We wish to show that they are equal. Suppose that $y \in \tau P$ but $y \notin \bigcup(\tau^{-1}s^* + P)$. Then there is an open neighbourhood N of y in $\mathbb{R}^n$ that lies entirely outside of $\bigcup(\tau^{-1}s^* + P)$. Since τP is convex and contains an open ball of $\mathbb{R}^n$, it is easily seen that $N \cap \tau P$ contains a non-empty open subset U. Since L^* is dense in $\mathbb{R}^n$, $L^* \cap U \neq \emptyset$. Any point $x \in L$ with $x^* \in L^* \cap U$ lies in $\Sigma_{\tau P}$ but not in $\Sigma_{\bigcup(\tau^{-1}s^*+P)}$, a contradiction. Thus $\tau P \subset \bigcup(\tau^{-1}s^* + P)$. A similar argument gives the reverse inclusion. □

Proposition 3.3. Let P be a closed bounded convex region in $\mathbb{R}^n$. Suppose that S is a finite subset of $\mathbb{R}^n$ such that the replicating condition (RC) holds for S and P. Let

$$X := \{x \in \Sigma_P \mid \|x\| \leqslant \tau^3 \mu\}$$

where $\mu := \max\{\|s\| \mid s \in S\}$. Then X is finite and $\Sigma_P = (S; X)^{\vdash}$.

Proof. The set X is finite by lemma 2.5. By proposition 3.2, $\Sigma_P = \bigcup_{s\in S} s \vdash \Sigma_P$. Let $x \in \Sigma_P$ be fixed and suppose that $\|x\| > \tau^3\mu$. We can now write $x = s \vdash y = T_s(y)$. We have

$$\begin{aligned}
x = \tau^2 s - \tau y &\Longrightarrow \tau y = \tau^2 s - x \\
&\Longrightarrow \tau \|y\| \leqslant \tau^2 \|s\| + \|x\| \\
&\Longrightarrow \|y\| \leqslant \tau\mu + \tau^{-1}\|x\| < (\tau^{-2} + \tau^{-1})\|x = \|x\|.
\end{aligned}$$

In this way we replace x by an element of smaller norm. Since by lemma 2.5 $\{z \in \Sigma_P \mid \|z\| < \|x\|\}$ is finite for each $M \geqslant 0$, after a finite number of repetitions of this argument we obtain a point $z \in X$ and a sequence $s_1, \ldots, s_k$ of elements of S with $x = T_{s_k} \ldots T_{s_1}(z)$. □

4. One-dimensional quasilattices

Consider the (R^*)-module $\mathcal{R}$. Let $a \in R$ and define

$$\Sigma(a) = \Sigma_{[0,a]}(\mathcal{R}) = \{x \in R \mid x' \in [0, a]\}$$

where $[0, a]$ is the closed interval with end points 0 and a. Since

$$\tau[0, a] = [0, a] \cup (\tau^{-1}a + [0, a])$$

we see that (RC) holds with $S = \{0, a'\}$ and by proposition 3.3

$$\Sigma(a) = (\{0, a'\}; X)^{\vdash}$$

with

$$X = \{x \in \Sigma(a) \mid \|x\| \leqslant \tau^3 \|a\|\}.$$

If $a = 0$ then $\Sigma = \{0\}$. The most important case is $a = 1$. Then a short calculation shows that $X = \{0, 1, -\tau, \tau^2, -\tau^3\}$. Since

$$-\tau = T_0(1) \qquad \tau^2 = T_0(-\tau) \qquad -\tau^3 = T_0(\tau^2)$$

we have:

Proposition 4.1. $\Sigma_{[0,1]}(R) = \{0, 1\}^{\vdash}$.

We have the following explicit description of $\Sigma_{[0,1]}(R)$.

Proposition 4.2.
(i) $\Sigma_{[0,1]}(R) = \{1\} \cup \{\lceil b\tau^{-1}\rceil + b\tau : b \in \mathbb{Z}\}$, where $\lceil\,\rceil$ is the roof function ($\lceil c\rceil$ is the least integer not less than c);
(ii) $(\{0, 1\} : \{0\})^{\vdash} = \{\lceil b\tau^{-1}\rceil + b\tau : b \in \mathbb{Z}\}$.
In particular, $\Sigma_{[0,1]}(R)$ and $(\{0, 1\} : \{0\})^{\vdash}$ are Delaunay sets.

Proof. $x \in \Sigma_{[0,1]}(R) \iff x = a + b\tau$, $0 \leqslant a + b\tau' \leqslant 1$, $a, b \in \mathbb{Z}$. With the exception $b = 0$, $a = 0, 1$, we have

$$\begin{aligned} 0 \leqslant a + b\tau' \leqslant 1 &\iff 0 < a + b\tau' < 1 \\ &\iff a > -b\tau' > a - 1 \\ &\iff \lceil b\tau^{-1}\rceil = a. \end{aligned}$$

This proves (i). Now we observe that $1 \vdash 1 = 1$ and $0 \vdash 1 = -\tau = 1 \vdash (1 \vdash 0)$. Thus in (i) we need the number 1 in the passive set only to generate 1 itself. Furthermore, if 1 is omitted from the passive set then it can no longer be generated. This follows from *inequalities* $\|k \vdash b\| > \|b\|$ *for* $k = 0, 1$ *provided that* $\|b\| > \tau^3$ *and inspection of the first* few values of generated in $(\{0, 1\} : \{0\})^{\vdash}$. □

Notation 4.3. Let L be an R module. Let $u, w \in L$, $v \neq w$. Then $\Sigma_{v,w} := \{v, w\}^{\vdash}$, $\Sigma_v := \{0, v\}^{\vdash}$. In particular, $\Sigma_1 \subset R$ is defined to be $\{0, 1\}^{\vdash}$. By proposition 4.1, $\Sigma_1 = \Sigma_{[0,1]}(R)$.

Proposition 4.4. Let L be an R-module and let $v, w \in L$, $v \neq w$. Then $\Sigma_{v,w} \simeq \Sigma_1$. In particular $\Sigma_{v,w}$ is a Delaunay set.

Proof. Consider the mapping $\phi : R \longrightarrow L$ defined by $\phi(r) = v + r(w - v)$. This maps $\{0, 1\}^{\vdash}$ bijectively onto $\{u, w\}^{\vdash}$. □

Thus in any quasilattice, the subset generated by any two distinct elements under $\vdash$ is a copy of Σ_1.

5. The cyclotomic integers $\mathbb{Z}[e^{2\pi i/5}]$

Let $\zeta := e^{2\pi i/5}$ and consider the cyclotomic field $\mathbb{Q}[\zeta]$ and its ring of integers $\mathbb{Z}[\zeta] = \sum_{j=0}^{4} \mathbb{Z}\zeta^j \subset \mathbb{C} \simeq \mathbb{R}^2$. We have $\dim_{\mathbb{Q}} \mathbb{Q}[\zeta] = \mathrm{rank}_{\mathbb{Z}} \mathbb{Z}[\zeta] = 4$, the basic relation among the powers of ζ being

$$0 = 1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4.$$

The Galois group $G = \mathrm{Gal}(\mathbb{Q}[\zeta]/\mathbb{Q}) \simeq (\mathbb{Z}/5\mathbb{Z})^\times$ is cyclic of order 4 with generator defined by $\zeta \longmapsto \zeta^2$. Since $\tau = -(\zeta^2 + \zeta^3)$ and $\tau' = -(\zeta + \zeta^4)$ we see that

$$\mathbb{Q}[\zeta] \cap \mathbb{R} = \mathbb{Q}[\tau] \qquad \mathbb{Z}[\zeta] \cap \mathbb{R} = \mathbb{Z}[\tau] = R$$

and * induces the automorphism $'$ on $\mathbb{Z}[\tau]$. Thus the pair $\mathcal{Z}[\zeta] = (\mathbb{Z}[\zeta], ^*)$ is an $(R, ^*)$-module of rank 2 in a natural way.

We let P_5 be the (solid) pentagon with vertices $S_5 := \{1, \zeta, \zeta^2, \zeta^3, \zeta^4\}$. Then $S_5^* = S_5$ and geometrically it is easy to see that $\tau P_5 = \bigcup_{s \in S} \tau^{-1} s^* + P_5$.

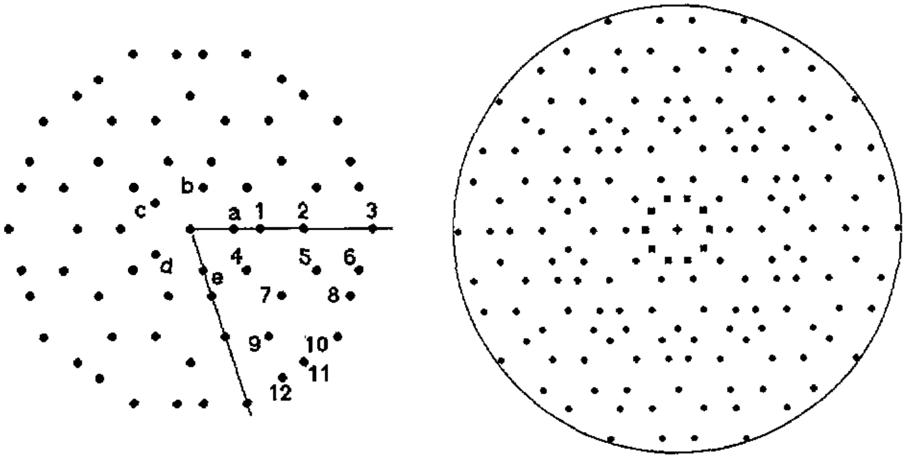

Figure 2. Quasilattices $\Sigma_{P_5}(\mathcal{Z}[\zeta])$ and $\Sigma_{P_{10}}(\mathcal{Z}[\zeta])$. The points labelled 1 through 10 are generated by the operators $T_a, \ldots, T_e$ acting on $\{0, a, b, c, d, e\}$ (see the table above proposition 5.1). The points of $\Sigma_{P_{10}}(\mathcal{Z}[\zeta])$ shown here (complete out to radius 7) are also obtained after two iterations of the operators T_k, $k \in S_{10}$, on the set S_{10}.

Thus $\Sigma_{P_5}(\mathcal{Z}[\zeta]) = (S_5; X)^\vdash$, where $X = \{x \in \Sigma_{P_5} \mid |x| \leqslant \tau^3\}$. Figure 2 illustrates the entire set of points of Σ_P with $|x| \leqslant \tau^3$. It is straightforward to see that in fact we need only the points $S_5 \cup \{0\}$ as passive generators. The following table shows the requisite calculations for the points indicated by the numbers 1 through 12.

(1) $T_e T_b^2(0) = \tau a$	(2) $T_a(0) = \tau^2 a$
(3) $T_b T_c(0) = \tau^3 a$	(4) $T_c^2(0) = -\tau c$
(5) $T_e(\tau d) = T_e T_c T_e^2(0) = T_e^2 T_c^2(0)$	(6) $T_a(c)$
(7) $T_a(b) = -\tau^2 c$	(8) $T_b^3 T_e^2(0)$
(9) $T_a^2 T_c^2(0)$	(10) $T_a T_e^2(0) = -\tau^3 c$
(11) $T_d^3 T_a^2(0)$	(12) $T_e(c)$.

Proposition 5.1. The set of points x of $\mathbb{Z}[\zeta]$ such that x^* lies in the closed pentagon with vertices $S = \{1, \zeta, \zeta^3, \zeta^4, \zeta^5\}$ is generated by the action of $T_1, \ldots, T_{\zeta^4}$ on the set $S_5 \cup \{0\}$.

Remark 5.2. The quasilattice $S_5^{\vdash}$ consists of all points of $\Sigma_{P_5}(\mathbb{Z}[\zeta])$ of the form $\sum a_j \zeta^j$, where $\sum a_j \equiv 1 (\text{mod } 5)$.

A word about terminology: mathematical crystals are not lattices, but have the underlying symmetry of lattices. For this reason we have called our algebraic objects quasilattices rather than quasicrystals. The points of Penrose tilings have a right to be called the prototypes of all algebraic quasicrystals; they have an underlying symmetry of quasilattices.

Next consider the decagon $P_{10} = P_5 \cup -P_5$ with vertices $S_{10} = S_5 \cup -S_5$. The set of points of $\Sigma_{P_{10}}(\mathcal{Z}[\zeta])$ with $|x| \leqslant \tau^3$ is illustrated in figure 2. In this case $S_{10} \cup \{0\}$ suffices as a set of passive generators for the active generators S_{10}. Moreover, we have

$$0 = (-\zeta^2) \vdash (1 \vdash (-\zeta^3))$$

so we do not need to include 0 as a passive generator.

Proposition 5.3. $\Sigma_{P_{10}}(\mathcal{Z}[\zeta]) = S_{10}^{\vdash}$.

6. Quasicrystal monoids

Up to now we have not paid much attention to the operators T_x that arise in quasicyrstal addition. In this section we study the algebraic structure of the monoid generated by these operators. As before, we begin with an R-module L and subsets $S, X \subset L$. From these we derive the set of points $(S; X)^{\vdash}$. This set is by definition generated from the passive set X by the repeated action of operators T_s, $s \in S$. We denote by $\mathcal{T}_S$ the *monoid* of all operators on L_F generated by the elements T_s, $s \in S$. Since the elements of $\mathcal{T}_S$ are affine-linear maps, the restriction of the operators of $\mathcal{T}_S$ to operators $L \longrightarrow L$ is faithful and we will be free to think of $\mathcal{T}_S$ as operators on L or L_F. We call these monoids, *monoids of quasicrystal operators.*

If $(s_k, \ldots, s_1)$ is a sequence in S then we have the *word* $\tilde{w} = (T_{s_k}, \ldots, T_{s_1})$ and its corresponding value $w = T_{s_k}, \ldots, T_{s_1} \in \mathcal{T}_S$.

Let L be a free R-module with basis B and let $\mathcal{T}_B$ be the corresponding monoid. If K is any R-module and S any subset of K with $\text{card}(S) = \text{card} B$ then there is a natural R-module map $\phi : L \longrightarrow K$ induced from any bijection of B to S, and then an induced monoid homomorphism $\tilde{\phi} : \mathcal{T}_B \longrightarrow \mathcal{T}_S$ with $\tilde{\phi} T_b = T_{\phi(b)}$, for all B. We see then that $\mathcal{T}_B$ is a free monoid of quasicrystal operators and depends only on the cardinality α of B. We will usually denote $\mathcal{T}_B$ by $\mathcal{F}_B$ or $\mathcal{F}_\alpha$.

Because of proposition 2.1(iii), if $\alpha > 1$ then there are relations in $\mathcal{F}_B$:

$$T_x T_y^2 = T_y T_x^2 \qquad \text{for all } x, y \in B. \tag{QCM}$$

The main result of this section is that $\mathcal{F}_B$ is completely described by (QCM).

It is interesting to begin with a q-version of a quasicrystal. For this purpose we let $\mathbb{Z}[q]$ be the polynomial algebra in the indeterminate q and form the free $\mathbb{Z}[q]$-module

$$\mathbb{Z}[q]_\alpha = \bigoplus_{j \in \alpha} \mathbb{Z}[q] x_j$$

where $\{x_j\}$ is some basis indexed by the cardinal α.

We define $\vdash_q$ on $\mathbb{Z}[q]$ by

$$x \vdash_q y = q^2 x - qy$$

and hence operators $\hat{T}_x$, $x \in \mathbb{Z}[q]_\alpha$, analagous to the operators T_x. In particular we may form the monoid $\mathcal{F}_\alpha(q)$ generated by the operators $\hat{T}_i := \hat{T}_{x_i}$, $i \in \alpha$. We are interested, of course, in the case that when q is specialized to τ, but it is interesting to note the cases when q is specialized to ± 1 which are directly related to lattices.

Propostion 6.1. $\mathcal{F}_\alpha(q)$ is the free monoid on the generators $\hat{T}_i$, $i \in \alpha$.

Proof.

$$\hat{T}_{i_1} \cdots \hat{T}_{i_k} u = q^2 x_{i_1} - q^3 x_{i_2} + q^4 x_{i_3} - \cdots + (-q)^{k+1} x_{i_k} + (-q)^k u \tag{6.2}$$

from which it is obvious that

$$\hat{T}_{i_1} \cdots \hat{T}_{i_k} = \hat{T}_{j_1} \cdots \hat{T}_{j_l} \iff k = l \text{ and } i_p = j_p \qquad \text{for all } p.$$

□

Proposition 6.3. Let L be the free R-module with basis B. Suppose that $(x, y, z, \ldots)$ and $(u, v, w, \ldots)$ are finite sequences of elements of B and suppose that

$$T_x T_y T_z \cdots = T_u T_v T_w \cdots . \tag{6.4}$$

Then:

(i) the lengths of the words on the left- and right-hand sides are equal;

(ii) using only the relations (QCM) the left-hand side can be transformed by repeated substitutions into the right-hand side.

Proof. We use induction on the length of the left-hand side of (6.4). Since the operators T_p are invertible as transformations on L_F, we can cancel equal terms off the left ends of our words whenever they appear. Note that this cancellation is used only as a matter of convenience in the argument.

Using equation (6.2) and replacing q by τ we obtain

$$x + (-\tau)y + (-\tau)^2 z + \cdots = u + (-\tau)v + (-\tau)^2 w + \cdots .$$

Applying the automorphism of R and setting $\mu = -\tau' = \tau^{-1} \in (0, 1)$ we have

$$x + \mu y + \mu^2 z + \cdots = u + \mu v + \mu^2 w + \cdots . \tag{6.5}$$

We observe that

$$\sum_{j=2}^{\infty} \mu^j = 1 \qquad \mu + \mu^2 = 1. \tag{6.6}$$

It is already immediate from equation (6.5) that if one side of (6.4) is empty then so is the other. We assume then that neither side is empty. Furthermore, if $x = u$ we can drop the terms T_x and T_u and reduce the length. Henceforth we assume that $x \neq u$. Then in view of (6.6) and the R-independence of the elements of B, we have from (6.5) that

$$v = x$$

and similarly

$$y = u \neq x$$

so (6.5) becomes

$$x + \mu y + \mu^2 z + \cdots = y + \mu x + \mu^2 w + \cdots. \tag{6.7}$$

This already excludes the possibility that either the left- or right-hand sides have length less than 3.

If the left-hand side has length 3 then from (6.6) we see that $z = y$ and

$$x = \mu x + \mu^2 w + \cdots$$

from which $w = x$ and the right-hand side also has length 3. Thus our original equation reads

$$T_x T_y T_y = T_y T_x T_x$$

which is part of (QCM). Similar considerations dispose of the case that the right-hand side has length 3.

We now suppose that both sides have length greater than 3 and write (6.5) as

$$x + \mu y + \mu^2 z + \mu^3 z_1 + \cdots = y + \mu x + \mu^2 w + \mu^3 w_1 + \cdots. \tag{6.8}$$

On the left-hand side we still need to make up $(1-\mu)y = \mu^2 y$. Since $\sum_{j\geqslant 4} \mu^j = \mu^2$ we need at least one of z or z_1 to be y. Similarly one of w or w_1 is equal to x.

Case 1 ($z = y$). In this case our relation reads

$$T_x T_y T_y T_{z_1} \cdots = T_y T_x T_w \ldots$$

and so by (QCM) we get

$$T_y T_x T_x T_{z_1} \cdots = T_y T_x T_w \ldots.$$

Cancelling reduces the length and we are done by induction.

Case 2 ($w = x$). This is similar to case 1.

Case 3 ($z \neq y, w \neq x$). We have

$$x + \mu y + \mu^2 z + \mu^3 y + \cdots = y + \mu x + \mu^2 w + \mu^3 x + \cdots .$$

If $z = x$ then $\mu + \mu^3 + \ldots < \mu + \sum_{i=3}^{\infty} \mu^i = 2\mu$ must exceed $1 + \mu^2$, which is false. Thus $z \neq x$, and similarly $w \neq y$. Now we claim that $w = z$, for if not then $\sum_{i=4}^{\infty} \mu^i = \mu^2$ must exceed μ^2, which is false. Thus

$$x + \mu y + \mu^2 z + \mu^3 y + \cdots = y + \mu x + \mu^2 z + \mu^3 x + \cdots . \tag{6.9}$$

To complete the content of y on the left-hand side of (6.9) we need $\mu + \mu^3 + \cdots = 1$ and so $\mu^3 + \cdots = \mu^2$.

Thus we either have a term $\mu^4 y$ or $\mu^5 y$ If there is no μ^4 term then we have

$$\mu^3 + \mu^5 + \cdots = \mu^2$$

and so we have either μ^6 so $\mu^3 + \mu^5 + \mu^6 = \mu^3 + \mu^4 = \mu^2$, or μ^7. (Otherwise, $\sum_{i=8}^{\infty} \mu^i = \mu^6$ shows that we cannot arrive at μ^2.) Repeating this argument we see that, due to finiteness of the left-hand side, we have to have a solution

$$\mu^3 + \mu^5 + \cdots + \mu^{2k-1} + \mu^{2k} = \mu^2 \qquad \text{for some } k.$$

Thus the left-hand side of (6.4) is

$$T_x T_y T_z T_y T_{z_1} T_y T_{z_2} T_y \ldots T_{z_m} T_y T_y T_{z_{m+1}} \ldots .$$

From (QCM) we have $T_{z_m} T_y T_y = T_y T_{z_m} T_{z_m}$, and this initiates a chain of reductions resulting in

$$T_x T_y T_y T_z^2 T_{z_1}^2 \ldots T_{z_m}^2 T_{z_{m+1}}^2 .$$

Finally, after one more replacement, we have transformed the left-hand side to $T_y T_x^2 T_z^2 \ldots$ and we can cancel terms from the left- and right-hand sides thereby reducing the length.□

Corollary 6.10. $\mathcal{F}_B$ is isomsorphic to the free monoid $\mathcal{M}$ with generators t_x, $x \in B$, and relation

$$t_x t_y^2 = t_y t_x^2 \qquad \text{for all } x, y \in B.$$

In particular $\mathcal{F}_B$ is embeddable in a group.

Proof. Let $f : \mathcal{M} \longrightarrow \mathcal{F}_B$ be the unique monoid homomorphism with $f(t_x) = T_x$. Suppose that $f(t_{x_1} \ldots t_{x_k}) = f(t_{y_1} \ldots t_{y_l})$, where $x_1, \ldots, x_k, y_1 \ldots, y_k \in B$. Then

$$T_{x_1} \ldots T_{x_k} = T_{y_1} \ldots T_{y_l}$$

and by using the relation $T_x T_y^2 = T_y T_x^2$ we may rewrite the left-hand side as the right-hand side. The same replacements in the t's show that $t_{x_1} \ldots t_{x_k} = t_{y_1} \ldots t_{y_l}$. □

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